Problem 1. Leader-follower game

a) Consider the two autonomous driving game from Lecture 1.

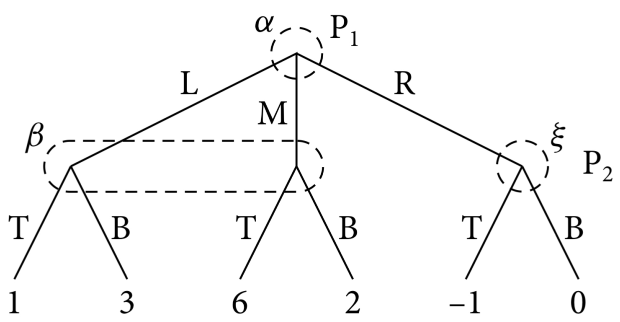
   (i) What is the security strategy of player 1? What would be the game outcome if both players play their corresponding security strategies?

   (ii) Compute the Stackelberg equilibria for two cases: player 1 being the leader, and player 2 being the leader. Verify that if either of the players act as a leader, its payoff won't be worse than that of her/his Nash equilibrium.

b) Now consider a zero-sum finite action game where there is no pure strategy Nash equilibrium. Prove that in this setting, the leader will always be worse off. (Note: in Lecture 05, we saw that in this case, the leader may no longer be better off playing its Stackelberg equilibrium strategy in comparison to its Nash equilibrium strategy as at least one of the assumptions of the theorem is not satisfied).

Problem 2. Subgame perfect behavioral equilibrium

Consider the feedback game below (see also Figure 7.4 of Hespanha book). Player 1 is a minimizer and player 2 is a maximizer.



a) Is this a perfect information game?

b) Formulate the game in matrix form. Verify that the game has several pure strategy Nash equilibria, find these equilibria. And verify that no pure strategy subgame perfect equilibrium exist.

c) Determine the behavioral strategy subgame perfect equilibrium of the game.

d) Verify that the value of the game corresponding to the behavioral strategy subgame perfect equilibrium is the same as the value of any of the pure strategy Nash equilibria. Could you have made this conclusion without computing the equilibria of the game?

Problem 3. Correlated equilibria and no-regret learning

Consider the identical interest game where players are utility maximizers:

$$\begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 2 & 1 & -4 \\ 1 & 0 & -1 \\ -4 & -1 & -2 \end{bmatrix} \end{array}$$

This game is taken from Figure 3 (i) of the following article [1]. The row player corresponds to player 1 and the column player corresponds to player 2.

a) Find the unique pure strategy Nash equilibrium of the game by removing strictly dominated actions.

b) Verify that the probability distribution D that assigns probability 1/3 to strategies corresponding to the diagonal entries is a coarse correlated equilibrium (CCE) of the game.

c) According to Lecture 01, rational players should not choose strictly dominated actions. Argue that a coarse correlated equilibrium may not satisfy the rationality model above.

d) Now, we consider player 1 implementing the multiplicative weight update algorithm, while player 2 plays her constant Nash equilibrium strategy. Let the sequence of played actions by player 1 be $\{a_t^1\}_{t=1}^T$. Based on the no-regret property of this algorithm, what should the empirical frequency $\sigma_\tau^1(a) = \frac{1}{\tau}\{|t \in [\tau] : a = a_t^1\}$, $\tau \in [T]$ of player 1 converge to?

e) Now, consider both players implementing the multiplicative weight update algorithm. What should the empirical frequency $\sigma_\tau(a) = \frac{1}{\tau}\{|t \in [\tau] : a_t = (a_t^1, a_t^2)\}$, $\tau \in [T]$ converge to? Can you conclude anything about the convergence of the actual sequence of played actions $\{a_t\}_{t=1}^T$?

Problem 4. Dynamic games: Zero-sum LQR

In Lecture 02, we proved that a zero-sum game has a mixed strategy Nash equilibrium, also referred to as saddle point equilibrium, and the game has a value. To this end, we leveraged our Lecture 01 result on the existence of mixed strategy Nash equilibria for general-sum games. Recall that a saddle point equilibrium has the property that

$$\max_{z\in\mathcal{Z}} \min_{y\in\mathcal{Y}} \; y^\top A z = \min_{y\in\mathcal{Y}} \max_{z\in\mathcal{Z}} \; y^\top A z,$$

with $\mathcal{Y} \subset \mathbb{R}^{m_1}$, $\mathcal{Z} \subset \mathbb{R}^{m_2}$ being simplexes and m_i denoting the number of actions of player i, $i \in \{1, 2\}$.

A more general result from convex optimization is that if a function $J : Y \times Z \to \mathbb{R}$ is convex in y and concave in z, with Y, Z being convex sets, we have:

$$\min_{y\in Y} \max_{z\in Z} J(y, z) = \max_{z} \min_{y} J(y, z). \tag{0.1}$$

Based on the above result, verify the stated result in Exercise 4.3 of Hespanha. Next, equipped with the above, we will tackle zero-sum linear quadratic games.

Consider a zero-sum discrete-time linear quadratic game with state dynamics for $k = 1, 2, ...$ given as

$$x_{k+1} = Ax_k + Bu_k + Ev_k,\; x \in \mathbb{R}^n, u \in \mathbb{R}^{m_u}, v \in \mathbb{R}^{m_v}.$$

Above, u_k, v_k are player 1 and 2 actions at time k, respectively. Player 1 aims to minimize the following cost function while player 2 aims to maximize it:

$$\sum_{k=1}^{K} \left(x_k^T Q x_k + u_k^T u_k \right) - \mu^2 v_k^T v_k.$$

Above, $Q \succeq 0$, and thus, $Q = C^T C$ for some $C \in \mathbb{R}^{p\times n}$, $p \leq n$ (from properties of positive semidefinite matrices). Furthermore, μ^2 can be considered as a constant that maps units of v_k to that of u_k (if players' have different actuation possibilities). The control interpretation of the formulation above is that player 1 aims to keep the output $y_k = Cx_k$ small with minimum energy (thus, penalizing $\|u_k\|_2^2$, while player 2 aims to keep the output large with minimum energy (note that player 2 is maximizing the cost).

a) Consider a pair of strategies (γ, σ), where $\gamma = (\gamma_1, \dots, \gamma_K)$, $\sigma = (\sigma_1, \dots, \sigma_K)$, $\gamma_t : X \to \mathbb{R}^{m_u}$, $\sigma_t : X \to \mathbb{R}^{m_v}$, $t = 1, 2, \dots, K$. Use backward induction to write the equations which a pure strategy subgame perfect equilibrium (γ^*, σ^*) must satisfy. *Hint:* Start the backward recursion with $V_{K+1}(x) = 0$.

b) Under which condition a pure strategy subgame perfect linear state feedback equilibrium exists?

References

[1] Yannick Viossat and Andriy Zapechelnyuk. No-regret dynamics and fictitious play. *Journal of Economic Theory*, 148(2):825–842, 2013.